Approximate analytic solutions for noise-induced coherent oscillations in autonomous nonlinear systems

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Noise-induced coherent oscillations in autonomous nonlinear systems, recently found in numerical studies by Gang *et al.* [Phys. Rev. Lett. **71**, 807 (1993)], are explained in simple terms by obtaining approximate analytical solutions in the weak-noise regime. The creation by noise of asymptotic collective oscillations in deterministic systems near saddle-node bifurcation points is understood as a process of generation of probability currents in effective monodimensional potentials. The knowledge of the functional dependence of the mean frequency on the parameters of the system gives insight into the mechanisms responsible for specific features of these resonancelike responses to noise. [S1063-651X(97)01712-1]

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I. INTRODUCTION

In the study of complex nonequilibrium systems, a common strategy is the elimination through a process of averaging of dynamically nonrelevant variables to reduce the systems to equivalent ones of only a few degrees of freedom. When there is spatial uniformity, the time evolution of these contracted systems is usually governed by ordinary differential equations. Nonlinearity in them can give rise to a rich dynamics in which limit cycle oscillations, bistability, multistability or chaos can be present. In the stochastic approach, fluctuations from the averages, coming from internal or external sources of noise, are considered, and in many cases a satisfactory description of the systems in terms of deterministic equations modified by stochastic forces is possible. In this framework, the interplay between noise and nonlinearity may result in highly nontrivial effects, such as noise-induced transitions and stochastic resonance. The ubiquitous character of fluctuations, and their traditional role in producing deleterious effects, justify the interest in studies that can reveal their ability to induce coherent behavior. In fact, there is a wide variety of physical contexts in which cooperative effects of noise and nonlinearity can be relevant, and possible practical applications of these studies range from the implementation of methods to control oscillating chemical reactions to the understanding of self-organization in biological processes or the improvement of the performance in laser systems.

Recent work in this field [1] has been dedicated to studying the influence of additive white noise on the coherent motion of bidimensional autonomous nonlinear systems inside saddle-node bifurcation regions. Numerical studies of corresponding systems of coupled Langevin equations revealed the existence of effects such as noise-induced frequency shift when the deterministic system has a limit cycle, and noise-induced coherent oscillations in the absence of a deterministic limit cycle. A resonancelike character in these responses to noise was detected, and this behavior in autonomous systems was compared with the phenomenon of stochastic resonance in driven nonlinear systems.

In this paper, approximate analytical results for the two models studied in Ref. [1] are presented. Starting from the Fokker-Planck equations, it is shown that in the weak-noise regime, probability distribution functions can be factorized and stationary solutions can be obtained. The drastic effects induced by noise are explained in simple terms by obtaining an effective mean frequency for the systems. Some of the most interesting results of Ref. [1] were obtained for values of noise widely out of the small-noise region; for example, the peak structure in the dependence of the frequency on the noise strength, and the disappearance of the preferred finite frequency, were found only for high noise intensities. It is shown here that a weak-noise approximation can give some clues to understanding these effects, and, consequently, can help in the design of models in which, by modifying the functional structure of the deterministic systems or the properties of the stochastic forces, these effects can be controlled.

The outline of the paper is as follows. Aspects of models that are especially relevant to the role played by noise in the dynamics are analyzed in Sec. II. In Sec. III, a treatment of the Fokker-Planck equations in a weak-noise approximation is given, and explicit solutions are presented. Finally, in Sec. IV some conclusions are summarized.

II. MODELS

Models 1 and 2, presented in Ref. [1] as sets of two coupled Langevin equations with additive Gaussian white-noise terms, have in common the nongradient character of the drift terms. In fact, the deterministic parts of both systems present, for a certain value of a parameter of control, a saddle-node bifurcation that gives rise to a stable nonuniform circular limit cycle. The potential conditions [2] are therefore not satisfied even in the case of equal noise strength in the two equations and, as a consequence, rotational probability flows can occur [3]. There is a difference in stability character: in a certain range of the parameter of control, model 1 is bistable, while model 2 is monostable; this is not relevant for the role played by noise in inducing asymptotic oscillations. Moreover, the similar functional structure of both models allows a parallel treatment and a common discussion, so we have opted to focus on model 1, while occasionally presenting results for model 2.

In polar coordinates, which are directly related to variables normal and tangential to the possible limit cycles [4], the Langevin equations for model 1 in the Stratonovich interpretation are written as

$$\dot{r}=r(1-r^2)+\cos(\theta)W_1(t)+\sin(\theta)W_2(t),$$
$$\dot{\theta}=b-r^2\cos(2\theta)-\frac{\sin\theta}{r}W_1(t)+\frac{\cos\theta}{r}W_2(t), \tag{1}$$

where $W_1(t)$ and $W_2(t)$ are uncorrelated Gaussian white-noise terms with moments

$$\langle W_i(t)\rangle=0,$$
$$\langle W_i(t)W_j(t')\rangle=D_i\delta_{ij}\delta(t-t'),\quad i,j=1,2. \tag{2}$$

For model 2 the equation that gives the time evolution of the radial variable is exactly the same, and there are only differences with respect to model 1 in the second equation. We then have, for model 2,

$$\dot{r}=r(1-r^2)+\cos(\theta)W_1(t)+\sin(\theta)W_2(t),$$
$$\dot{\theta}=b-r\cos\theta-\frac{\sin\theta}{r}W_1(t)+\frac{\cos\theta}{r}W_2(t). \tag{3}$$

The corresponding Fokker-Planck equations for the probability densities $P(r,\theta,t)$ are readily obtained. Restricting ourselves to the particular case studied in Ref. [1], $D_1=D_2\equiv D$, we have

$$\frac{\partial P(r,\theta,t)}{\partial t}=-\frac{\partial(D_rP)}{\partial r}-\frac{\partial(D_\theta P)}{\partial\theta}+\frac{\partial^2(D_{rr}P)}{\partial r^2}+\frac{\partial^2(D_{\theta\theta}P)}{\partial\theta^2}, \tag{4}$$

being, for model 1,

$$D_r=r(1-r^2)+\frac{D}{2r},$$
$$D_\theta=b-r^2\cos(2\theta), \tag{5}$$

and

$$D_{rr}=\frac{D}{2},$$
$$D_{\theta\theta}=\frac{D}{2}\frac{1}{r^2}. \tag{6}$$

We see that, even in this particular case, the system, because of the nongradient character of its deterministic part, does not meet the potential conditions, and therefore it is not trivial to find the stationary solution. Instead of trying to solve the bidimensional Fokker-Planck equation, we have obtained from it the following set of simpler Langevin equations for an equivalent system [2]:

$$\dot{r}=r(1-r^2)+\frac{D}{2r}+\left(\frac{D}{2}\right)^{1/2}\Gamma_1(t),$$
$$\dot{\theta}=b-r^2\cos(2\theta)+\left(\frac{D}{2}\right)^{1/2}\frac{1}{r}\Gamma_2(t), \tag{7}$$

where $\Gamma_1(t)$ and $\Gamma_2(t)$ are again uncorrelated Gaussian white-noise terms:

$$\langle\Gamma_i(t)\rangle=0,$$
$$\langle\Gamma_i(t)\Gamma_j(t')\rangle=2\delta_{ij}\delta(t-t'),\quad i,j=1,2. \tag{8}$$

It must be noted that only in the case when $D_1=D_2$ is this simplified description possible, and in fact we will see that to leave this restriction can lead to a different qualitative behavior in the system.

A key point in the analysis is to realize that, in both models, the equation for the angular variable for a fixed value of r corresponds to a process of diffusion in a biased periodic potential with different amplitude and periodicity in each case. Another important feature of both models is that, in the alternative description that we have found, the first equation has additive white noise and does not depend on θ, so the probability density for r reaches its stationary state independently of the time evolution of θ. It can be seen that the spurious drift term that comes from interpreting in the Stratonovich sense the multiplicative character of the noise terms in Eq. (1) has been incorporated in an effective way in the deterministic dynamics in this alternative description, altering the size of the limit cycle if this is present, and making the unstable equilibrium point at the origin disappear. The corresponding Fokker-Planck equation for r with the boundary condition of zero probability current, imposed by the reflecting walls at the extremes of the radial range, can easily be solved to obtain the distribution function for the stationary state $P_{SS}(r)$ [3],

$$P_{SS}(r)=\frac{2r}{N}e^{-(r^2-1)^2/2D}, \tag{9}$$

where the normalization constant N is given by

$$N=\sqrt{2D}\int_{-1/\sqrt{2D}}^{\infty}e^{-y^2}dy=\left(\frac{\pi D}{2}\right)^{1/2}\operatorname{erfc}\left(-\frac{1}{\sqrt{2D}}\right), \tag{10}$$

$\operatorname{erfc}(x)$ being the complementary error function [5].

This probability density reaches its maximum value for

$$r_{\max}=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+2D}}, \tag{11}$$

and has a variance given by [3]

$$\langle\Delta r^2\rangle=\langle r^2\rangle-\langle r\rangle^2=1+\frac{D}{N}e^{-1/2D}-\left[\frac{(2D)^{3/4}}{N}e^{-1/2D}Z_2\left(-\frac{1}{2\sqrt{2D}}\right)\right]^2, \tag{12}$$

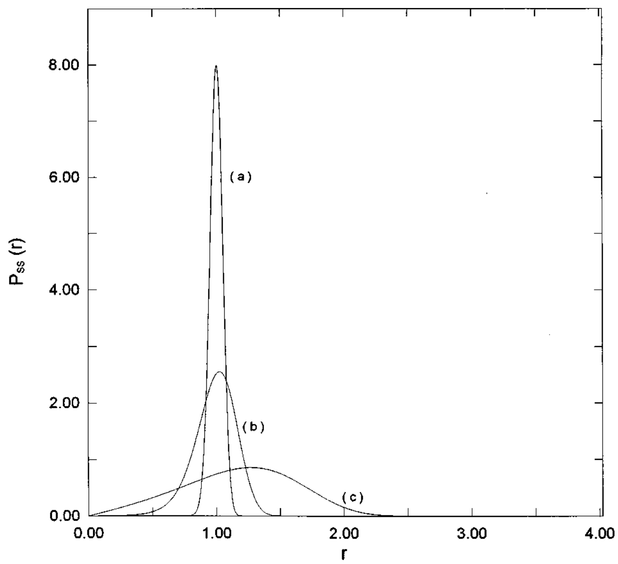

FIG. 1. Stationary probability density for the radial variable $P_{SS}(r)$, for $D=10^{-2}$ (a), $D=0.1$ (b), and $D=2$ (c).

where $Z_2(x)$ is given in terms of the modified Bessel functions of fractional order [5] by

$$Z_2(x)=x^{3/2}e^{2x^2}[K_{3/4}(2x^2)-K_{1/4}(2x^2)] \quad \text{for} \quad x>0,$$

and

$$Z_2(x)=|x|^{3/2}e^{2x^2}[K_{3/4}(2x^2)+K_{1/4}(2x^2)+\pi\sqrt{2}I_{3/4}(2x^2) + \pi\sqrt{2}I_{1/4}(2x^2)] \quad \text{for} \quad x<0. \tag{13}$$

The results for the variable r in model 2 are exactly the same.

In Fig. 1, $P_{SS}(r)$ is represented for different values of the noise strength D. For small noise ($D\ll 1$), the distribution function is very sharp, so we can conclude that in the stationary state the system is practically located in $r_{\max}$. As D increases, the maximum of the function is shifted to higher values of r, and the distribution becomes wider. Therefore the system is no longer located; instead it explores a larger range of values of the radial coordinate around $r_{\max}$. It can also be seen that although values of r smaller than 1 are possible for the system, any increase in D produces a net shift of probability to larger values of r because of the asymmetry of the distribution.

In Sec. III we present a method to obtain approximate expressions for the stationary probability distribution function $P_{SS}(r,\theta)$ which are valid in the small-noise regime. By now, we have obtained analytical solutions for

$$P_{SS}(r)=\int_0^{2\pi}P_{SS}(r,\theta)d\theta \tag{14}$$

which are valid for any value of the noise intensity.

III. WEAK-NOISE APPROXIMATION

A. Stationary probability distribution function

It can be shown [3] that for small values of the noise intensity, $D\ll 1$, the variance of $P_{SS}(r)$, given by Eq. (12), can be approximated by

$$\langle \Delta r^2\rangle = \langle r^2\rangle - \langle r\rangle^2 = \frac{D}{4}+O(D^2). \tag{15}$$

Therefore, in the weak-noise regime, and independently of the relative magnitude of the time scales involved in each equation, we can, as a first-order approximation, replace r by its maximum probability value in the equation that gives the time evolution of θ, and consequently, factorize the probability distribution function corresponding to the stationary state

$$P_{SS}(r,\theta)\simeq P_{SS}(r)P_{SS}(\theta;r_{\max}). \tag{16}$$

Of course, higher-order corrections must take into account the role of the radial fluctuations in the equation for the angular variable. In particular, these fluctuations become especially relevant where the phase motion is slowed down. Approximations similar to this one are frequently made in the study of nonlinear self-excited oscillations in radio engineering [3]. It is worthwhile to point out that a linearization in r is possible, but it is not neccessary because we have the exact solution, and that, in spite of the small intensity of noise, a linearization in θ is not possible because the effects we want to explain are directly related to a change in stability in the angular variable, which, given the parameters chosen for the system, can even be induced by small noise. For the same reason, a nonequilibrium potential that can explain in a unified way the qualitatively different behaviors induced by noise in the asymptotic dynamics cannot be found. On the other hand, the similarity in the time scales in both equations prevents the application of any adiabatic approximation or slaving principle. The results obtained will show that with this approximation we focus on the characteristics of the system that are relevant to give an explanation for the findings of Ref. [1].

The resulting effective equation for θ, which of course is only valid as a tool to calculate the stationary solution, is

$$\dot{\theta}=b-\tfrac{1}{2}(1+\sqrt{1+2D})\cos(2\theta)+\left(\frac{D}{1+\sqrt{1+2D}}\right)^{1/2}\Gamma(t), \tag{17}$$

with

$$\langle\Gamma(t)\rangle=0, \qquad \langle\Gamma(t)\Gamma(t')\rangle=2\delta(t-t'). \tag{18}$$

It is interesting to note that equations very similar to this one, and almost identical to the one corresponding to model 2, have appeared in very different physical contexts such as nonlinear self-excited oscillations in radio engineering [3], quantum noise in ring laser gyros [6], or thermal noise in Josephson junctions in the high-friction regime [7]. There is also a similarity between models 1 and 2, and the equations for the phase and intensity of a laser with a squeezed reservoir [8]. For that laser system, the deterministic dynamics above threshold is given by a gradient system with an attracting set, and the lack of detailed balance is caused by different noise intensities in the two Langevin equations.

From Eqs. (17) and (18), the Fokker-Planck equation is easily obtained, and, in terms of the probability current $G(\theta,t)$, a concept that will be useful to discuss detailed balance and calculate the mean frequency $\langle \dot{\theta} \rangle$, it can be written as

$$\frac{\partial P(\theta,t)}{\partial t}+\frac{\partial G(\theta,t)}{\partial \theta}=0, \tag{19}$$

with

$$G(\theta,t)=[b-f(D)\cos(2\theta)]P(\theta,t)-g(D)\frac{\partial P(\theta,t)}{\partial \theta}, \tag{20}$$

where we have defined $f(D)$ as

$$f(D)=\tfrac{1}{2}(1+\sqrt{1+2D}), \tag{21}$$

and $g(D)$ as

$$g(D)=\frac{D}{1+\sqrt{1+2D}}. \tag{22}$$

Given the angular character of the variable θ, and the fact that full rotations cannot be distinguished, periodic boundary conditions are added:

$$P(\theta,t)=P(\theta+2\pi,t). \tag{23}$$

The stationary solution is readily obtained:

$$P_{\mathrm{SS}}(\theta;r_{\max})=\frac{1}{N_1}e^{(b/g)\theta-(f/2g)\sin(2\theta)}\times\int_{\theta}^{\theta+2\pi}e^{-(b/g)\psi+(f/2g)\sin(2\psi)}d\psi, \tag{24}$$

where the normalization constant N_1 is given by

$$N_1=4\pi^2e^{-b\pi/g}\cosh(b\pi/2g)|I_{ib/2g}(f/2g)|^2, \tag{25}$$

and $I_{i\mu}(z)$ being the modified Bessel functions of imaginary order [5].

The corresponding results for model 2 are

$$P_{\mathrm{SS}}(\theta;r_{\max})=\frac{1}{N_2}e^{(b/g)\theta-(f^{1/2}/g)\sin\theta}\times\int_{\theta}^{\theta+2\pi}e^{-(b/g)\psi+(f^{1/2}/g)\sin\psi}d\psi, \tag{26}$$

with

$$N_2=4\pi^2e^{-b\pi/g}|I_{ib/g}(f^{1/2}/g)|^2. \tag{27}$$

In Fig. 2, the probability distribution function $P_{\mathrm{SS}}(r,\theta)$ for model 1 is represented for different values of the parameter b, and for different values of the noise strength, D. For $b>1$, it can be seen that the probability concentrates around the limit cycle that exists in the deterministic system, and that, by increasing D, the system becomes less located at the maxima of the distribution, so the nonuniformity of the limit cycle is attenuated when the noise is increased. The effect of the spurious drift term is hardly noticeable given the small values of the noise intensities. For $b<1$, which corresponds to the existence of a couple of stationary solutions for the asymptotic dynamics of the deterministic system, it can be seen that, for a certain value of D, there is a nonzero probability of finding the system in any point of a closed trajectory in the configuration space. There is again a spreading of probability for increasing D. The results for the mean frequency $\langle \dot{\theta} \rangle$ will confirm that in this case the periodic attractor is recovered.

The corresponding results for model 2 are presented in Fig. 3. The only significant difference between both models is that because of the monostable character of model 2, higher values of D are needed to generate rotational probability flows. This can be checked by analyzing the dependence of the mean frequency on noise in each model.

When these results are compared with the stationary distribution functions found in other studies about the influence of noise on systems with limit cycles [9,10], one main difference stands out: instead of the crater-shaped distribution with smooth changes of probability along the ridge that covers the deterministic trajectory, typical of systems with separability of variables and uniformity along the cycle, we have here that, because of the strongly nonuniform character of the cycles that appear via saddle-node bifurcations [11], the probability concentrates at two peaks centered at the most probable positions. Another characteristic of models 1 and 2 is that, although there is a coupling of variables, the time evolution of the radial variable does not depend on the angular variable, and, as a consequence, the transverse width of the probability distribution is the same along the limit cycle. This is in contrast to the features of a different model recently studied in Ref. [12]. There, nonuniformity and coupling between radial and angular motions were considered for systems with a stable limit cycle far from a Hopf bifurcation point, and a relation between the velocity along the cycle and the width of the distribution in the transverse direction was obtained.

Additional information about the asymptotic behavior of the system can be obtained by studying the effective mean frequency. This is the aim of Sec. III B.

B. Mean frequency in the stationary state

The mean frequency of the system is obtained averaging the effective Langevin equation for θ with the stationary probability distribution function. Given the zero mean value of the noise term, we obtain

$$\langle \dot{\theta} \rangle=\langle b-f\cos(2\theta)\rangle, \tag{28}$$

and it can easily be shown that, in terms of the probability current in the stationary state, this mean frequency can be written as

$$\langle \dot{\theta} \rangle=2\pi G_{\mathrm{SS}}. \tag{29}$$

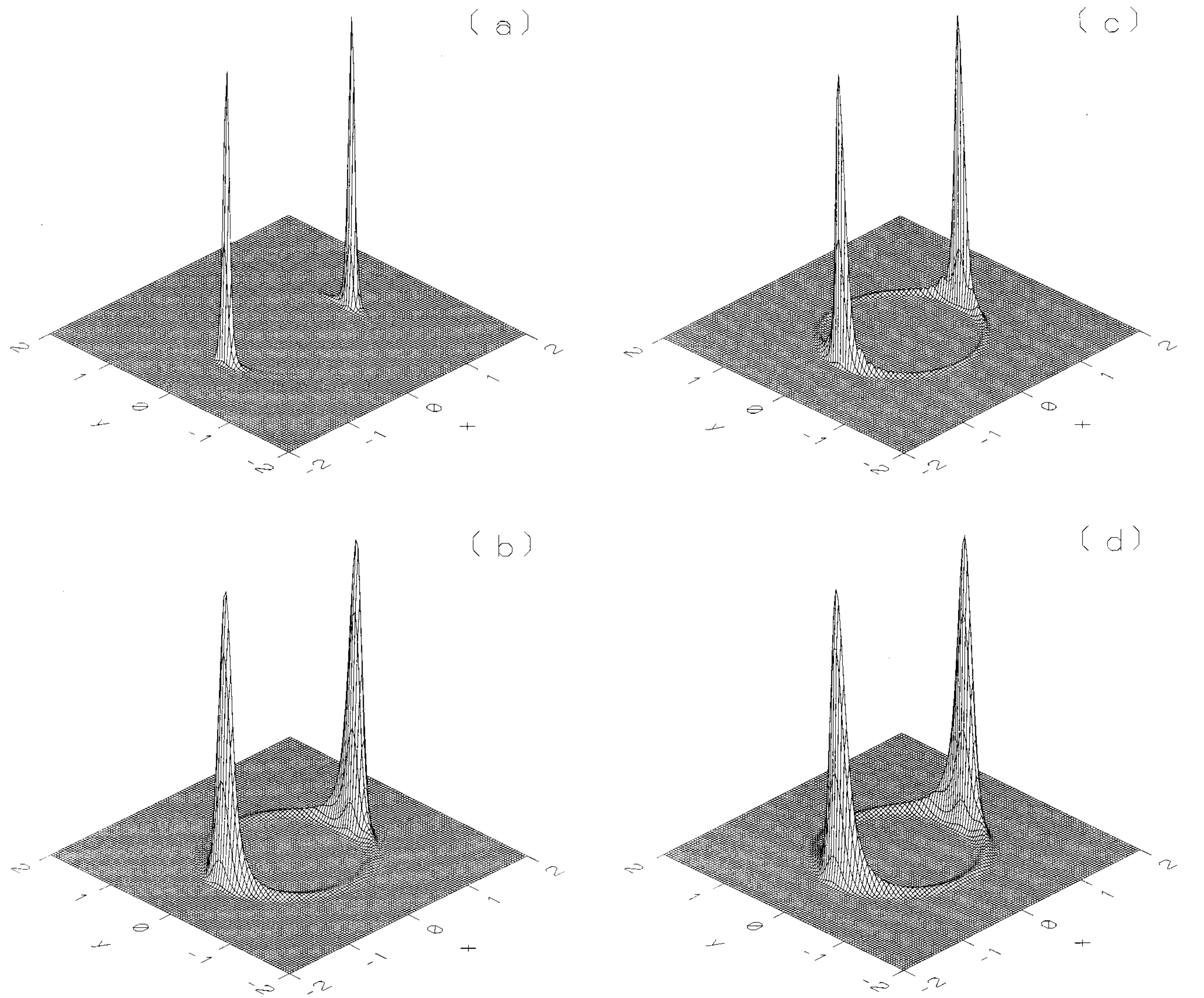

FIG. 2. Stationary probability density $P_{SS}(r,\theta)$, for model 1 for $b=0.99$ and $D=10^{-3}$ (a), $b=0.99$ and $D=10^{-2}$ (b), $b=1.01$ and $D=10^{-3}$ (c), and $b=1.01$ and $D=10^{-2}$ (d).

Therefore, any nonzero value for $\langle \dot{\theta} \rangle$ implies the existence of a probability current in the stationary state, and consequently the lack of detailed balance in the system.

Once the probability distribution function $P_{SS}(\theta; r_{\max})$ is known, G_{SS} is easily obtained, and consequently $\langle \dot{\theta} \rangle$

$$\langle \dot{\theta} \rangle = \frac{g}{\pi} \frac{\sinh(b\pi/g)}{\cosh(b\pi/2g)} |I_{ib/2g}(f/2g)|^{-2}. \qquad (30)$$

The corresponding result for model 2 is

$$\langle \dot{\theta} \rangle = \frac{g}{\pi} \sinh(b\pi/g) |I_{ib/g}(f^{1/2}/g)|^{-2}. \qquad (31)$$

In Fig. 4, $\langle \dot{\theta} \rangle$ for model 1 is represented for different values of b and D. It can be seen that the system is locked or unlocked depending not only on b, but also on D. For $b > f(D)$, which for the deterministic system corresponds to unlocking and a spectrum characterized by sharp lines at the fundamental frequency and its harmonics, we have obtained for the stochastic system that increasing values of D induce larger mean frequencies. In the limit $D \rightarrow 0$, the mean frequency tends to the mean frequency of the limit cycle. For $b < f(D)$, which corresponds to locking in the deterministic system, we see that the unlocking of the stochastic system takes place for values of D larger than a threshold value that depends on the parameters of the system. Therefore it can be said that noise causes the nonreduced system to anticipate the bifurcation. Subsequent increases in D result in higher values for $\langle \dot{\theta} \rangle$. It is important to notice that the threshold value depends on the relative magnitude of b and $f(D)$, higher values of D being necessary to unlock the system when $b/f(D)$ decreases. From this analysis the twofold effect of noise in the system is clear: on one hand, it tends, depending on the parameters, to unlock and/or increase the frequency; this can be seen by artificially giving a fixed value to $b/f(D)$. On the other hand, there is the opposite tendency coming from the functional dependence of f on D. This second aspect is hardly relevant in the small noise regime, but it can help to understand at least qualitatively the

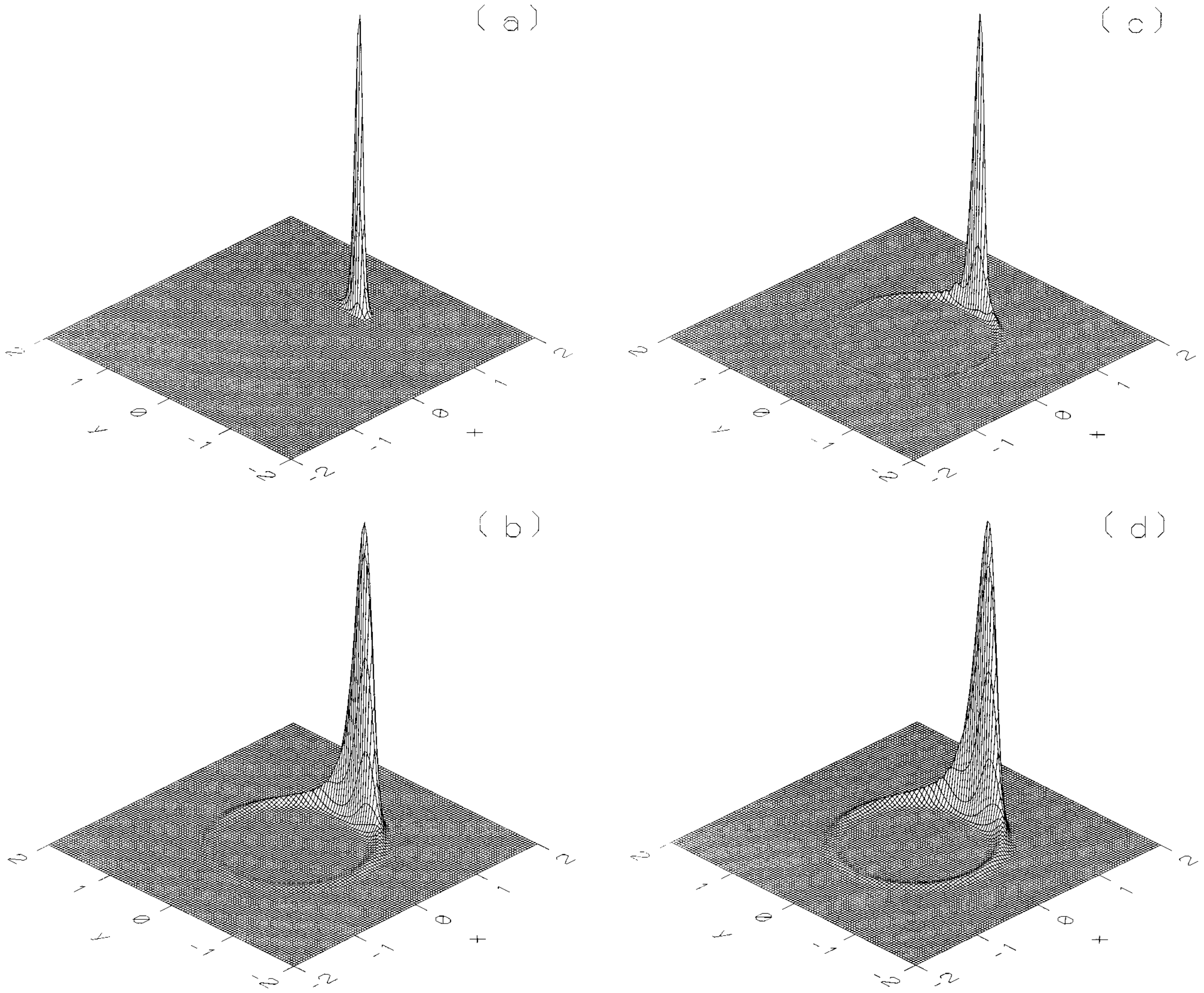

FIG. 3. Stationary probability density $P_{SS}(r,\theta)$, for model 2 for $b=0.99$ and $D=10^{-3}$ (a), $b=0.99$ and $D=10^{-2}$ (b), $b=1.$ and $D=10^{-3}$ (c), and $b=1.$ and $D=10^{-2}$ (d).

behavior for high noise. It must be emphasized that a stronger dependence of f on D can magnify the influence of this aspect on the dynamics.

For model 2, the effective equation for the variable θ is the same as the equation for the phase of the beat signal generated by a ring laser gyro. In Ref. [6] the spectrum for this parallel model was obtained analytically for $b \gg 1$, and also for parameters corresponding to locking; and numerically for the more interesting region $b \simeq 1$, in which there is a noise threshold value for unlocking, and that corresponds to the bifurcation region of models 1 and 2. For small noise these spectra have the same qualitative features as the ones obtained in Ref. [1]: basically, when noise is increased there is a shift of the peak frequency to higher values, and an increase in the width of the peak. For certain values of the parameters, $D \to 0$ and $b > 1$, higher harmonics can be seen. For very high noise the mean frequency would reach a limit value given by the bias b: the noise makes the existence of the periodic potential irrelevant and causes the bias to be the dominant feature. In Fig. 5, the mean frequency for model 2 is represented for different values of b and D.

We do not have analytical results for the high-noise re-

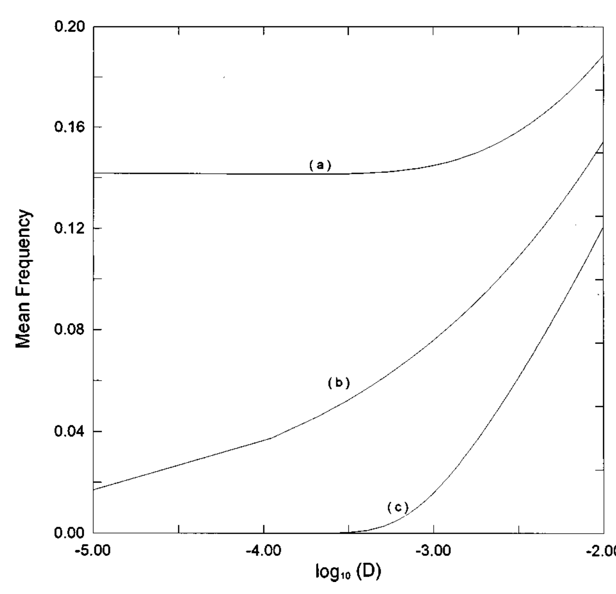

FIG. 4. Mean frequency $\langle \dot{\theta} \rangle$ for model 1 against $\log_{10} D$ for $b=1.01$ (a), $b=1.$ (b), and $b=0.99$ (c).

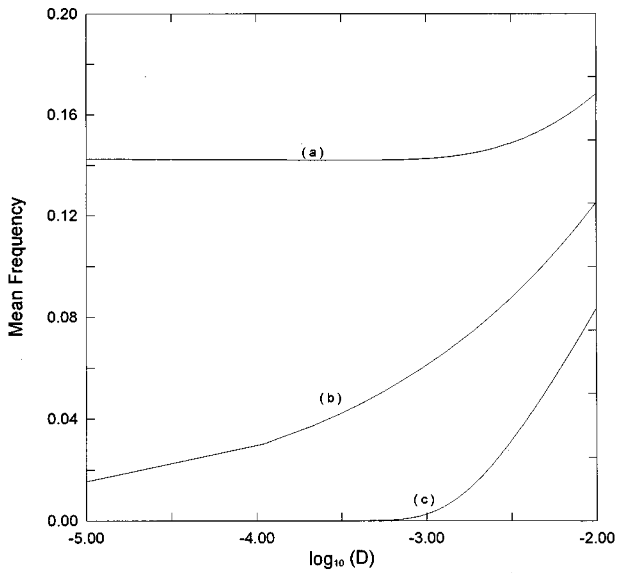

FIG. 5. Same as Fig. 4, for model 2.

gime, so we cannot explain the peak structure in the dependence of the frequency on D, and the disappearance of the coherent oscillations. Nevertheless, the physical picture obtained for small noise give us some clues to speculate about the qualitative behavior of the system for high D. As can be seen in Fig. 1, when D is increased, the system explores in its stationary state a larger range in the radial variable, the region of maximum probability being shifted to higher values. In this case we cannot replace r in the Langevin equation for θ by its maximum probability value, since the width of the radial distribution cannot be neglected, as can be seen studying the variance in Eq. (12). Studying the system of equations (7), which are exact for any value of the noise intensity, provides us with some ideas to conjecture which mechanisms are responsible for the nontrivial behavior of the system. First of all, we see that near r_{max}, the time evolution of the two variables, radial and angular, is slow as compared to the evolution for smaller r: the system spends more time near the asymptotic value of the deterministic radial equation, and any increase in r implies a higher amplitude of the effective periodic potential in the angular variable, and consequently larger barriers to the generation of probability currents. Therefore, we can think of two regions of maximum probability, centered on r_{max} and the two corresponding values of θ, from which noise induced excursions take place. Obviously the increase of r_{max} gives rise to a shift in the maximum probability values for θ: this explains the changing orientation of the line of maxima observed in Fig. 4 of Ref. [1]. In this way we can understand how the tendency to generate rotational probability flows, and increase the mean frequency that comes from the stochastic forces, can be counterbalanced by the larger localization of the system for higher values of r. Second, it is clear from Eqs. (7) that when the system reaches the region of small r, it experiences large uncorrelated jumps in θ that favor the exchange of probability through the central region and lead to the loss of coherence. This qualitative picture allows us to understand the topological changes in the probability distributions detected in Ref. [1] when D is increased. In effect, for sufficiently large noise, exchanges of probability between the two basins that correspond to the regions of maximum probability take place mainly through the region of small r rather than through the circle, and consequently the stochastic counterpart of the limit cycle, the circle hump, disappears. The peak structure in the dependence of the frequency on D and the disappearance of the preferred frequency can also be qualitatively understood: opposed to the shift to higher values of the frequency, which can be understood through the increase in the mean frequency caused by noise, there is the increasingly incoherent behavior that enlarges the width of the peak in the spectrum, lowers the value of the peak frequency, and eventually gives rise to the disapearance of the preferred frequency. It must be emphasized that whereas the increase in the frequency with D is a consequence of the nonlinear character of the system: noise tends to unlock the system by favoring diffusion in the effective biased periodic potential; the lowering of the peak frequency and the suppression of the coherent oscillation can also be detected in linear systems since they are a consequence of the loss of coherence. To check the validity of these ideas, let us consider the following system, which is directly related to the conjectured mechanism of loss of coherence for models 1 and 2, and can also represent the time evolution of the angular variable of a uniform limit cycle when noise is added,

$$\dot{\theta}=b+D^{1/2}\Gamma(t), \tag{32}$$

with

$$\begin{aligned} \langle\Gamma(t)\rangle&=0,\\ \langle\Gamma(t)\Gamma(t')\rangle&=2\delta(t-t'). \end{aligned} \tag{33}$$

In this case the mean frequency does not depend on noise, $\langle\dot{\theta}\rangle=b$, and the spectrum, which was obtained analytically in Ref. [6], reveals the existence of a preferred frequency that decreases with increasing D, and vanishes above a noise threshold that depends on the parameter b. Therefore it is clear that in this last regime the dynamics is dominated by the random term. This gives arguments of plausibility to the conjectures previously made for models 1 and 2 for high noise.

In this framework it is also possible to understand why the resonancelike behavior is not seen for small values of b [1]: the value of D needed to generate the probability current is so high that it prevents the appearence of any preferred frequency in the spectrum.

IV. CONCLUSIONS

The main conclusion of this work is the idea that the nontrivial effects found in the dynamics of the systems studied are a consequence of the interplay between the tendency of noise to unlock the systems and/or increase their frequency, and the loss of coherence also caused by noise that tends to diminish the frequency and destroy the collective oscillations.

The mechanism of recovery of the limit cycle when it is not present in deterministic systems is quite clear in the weak-noise approximation employed here: additive Gaussian white noise can induce a probability current in a one-dimensional system with a periodic potential modified by a bias term (the term linked to the parameter b in models 1 and 2). In this framework, the system is unlocked when the noise intensity reaches a certain threshold, and any additional increase in noise gives rise to an increase in frequency, and also to a spreading of the peaks in the spectrum. In terms of the bidimensional system, the unlocking is interpreted as allowed by the specific character of the bifurcation that gives rise to the cycle. In effect, as opposed to a Hopf bifurcation in which a limit cycle becomes smaller and finally collapses to a fixed point, in a saddle-node bifurcation the limit cycle becomes slower and slower and eventually vanishes at a finite trajectory, allowing noise to activate a circulation in the structure of the attractor.

The condition of equal noise strengths in the two equations of each model is a key point in having coupled variables with a deterministic dynamics for one of them, r, that depends on noise but does not depend on the second variable. This fact clarifies the mechanism behind the drastic effects of noise in the systems studied, and in a certain extent allows the control by noise of the time evolution of analog systems.

The resemblance of these results to the phenomenon of stochastic resonance is clear. Nevertheless, it must be emphasized that in models 1 and 2 of Ref. [1] the peak structure in the signal-to-noise ratio as a function of noise is basically a consequence of the definition of this ratio in terms of the frequency, since, as the widths of the peaks in the spectrum become larger for higher D, it is the peak structure in the frequency that is the dominant feature. The most typical scenario for stochastic resonance, the driven quartic double well in the high-friction regime and with Gaussian white noise, can be converted in an autonomous system by embedding it in a bidimensional system with the angle variable having a trivial time evolution [13]. In this bidimensional system with a noninvertible diffusion matrix, which as models 1 and 2, has nongradient drift terms, probability currents can exist, and, because of the nonlinearity of the problem, in addition to the fundamental frequency, sharp lines corresponding to the higher harmonics can also be seen in the spectrum. However, there is no noise-induced shifts of these frequencies as they are fixed at the frequency of the driving term and its harmonics, so the peak structure in the signal-to-noise ratio as a function of noise has therefore an origin different from that responsible for the findings of Ref. [1].

As we mentioned in Sec. I, sets of equations corresponding to models 1 and 2 can appear in the description of complex systems in which a process of averaging over dynamically nonrelevant variables has reduced the dimensionality of the problem. The nongradient character of the drift terms, which can have its physical origin in the averaging over variables whose dynamics are not time reversal invariant, is a neccessary condition for the appearance of limit cycles in the secular motion and also for the appearance of irreversible circulations in the stochastic system when the noise intensities are the same in the two equations of the reduced system. As a lack of detailed balance can also be reached in a bidimensional gradient system by working with noise terms with different characteristics for each variable, which is possible from a physical point of view if there is some control over external sources of noise, it would be interesting to analyze if noise can induce the stochastic counterpart of a limit cycle in a system whose deterministic dynamics cannot present periodic attractors.

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